Linear Algebra & Geometry

USOS Id: 1120-IN000-ISA-0114

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LECTURE 1

Basic terminology.

Complex numbers.

Sets and operations on sets

A *set* is an unordered collection of distinct objects taken from some "mother set" X usually called the universal set. The objects constituting a set are referred to as its *elements*. We will use $x \in S$ to denote that "x is an element of S" and $x \notin S$ to denote that it is not.

We enclose in curly brackets elements of a set, e.g. {1, 2} denotes the set whose elements are 1 and 2. Infinite sets must be described differently.

Example: $S = \{a, b, c, d\}$, where a, b, c, d represent themselves, i.e. letters of alphabet. Clearly, $a \in S$ and $e \notin S$. Be careful, though, if we use letters as symbols representing other objects, like numbers, it may happen that a and e represent the same thing and thus e does belong to S.

The set with no elements is called the empty set and is denoted by $\emptyset$. Remember that the empty set is not "nothing", hence $\{\emptyset\}$ is not the same as $\emptyset$. On the other hand, {} is the same as $\emptyset$.

If A and B are sets (of elements from some X) then we say that A *is a subset of* B and write $A \subseteq B$ if every element of A is also an element of B. In symbols $A \subseteq B$ *iff* $(\forall x \in X)(x \in \mathrm{A} \Rightarrow x \in \mathrm{B})$.

In this terminology, sets under consideration should be subsets of some set X.

We say that two sets A and B are *equal* if they consist of the same elements. We write then A=B. Clearly A=B if and only if A⊆ B and B⊆ A. In symbols A=B *iff* $(\forall x \in X)(x \in \mathrm{A} \Leftrightarrow x \in \mathrm{B})$.

Given two subsets A and B of X we construct some new sets:

- the *union* of A and B, $A \cup B = \{x \in X | x \in A \vee x \in B\}$
- the *intersection* of A and B, $A \cap B = \{x \in X | x \in A \wedge x \in B\}$
- the *difference* (or *relative complement*) of A and B, $A \setminus B = \{x \in X | x \in A \wedge x \notin B\}$
- the *(absolute) complement* of A, $A' = \{x \in X | x \notin A\}$. This is one reason why we should declare the universal set X beforehand, otherwise the symbol A' is ambiguous.

In each case, the resulting set is also a subset of X.

The *Cartesian product* of two subsets of X, $A \times B$, is a different story in that it may not be a subset of X:

$A \times B$ is the set of all *ordered pairs* (x,y) where $x \in A$ and $y \in B$. The *ordered pair* (x,y) differs from a two-element set {x,y} in that $(x,y) \neq (y,x)$ (unless, of course, x=y). Hence (x,x) is a proper ordered pair while the symbol {x,x}, even though legal, does NOT denote a two-element set, {x,x} = {x}, so {x,x} is a sloppy, redundant notation for a one-element set {x}. This is sometimes unavoidable – for example, when we write things like "$\{x_1,x_2\}$ is the set of roots of a quadratic polynomial" we do not know, in general, whether $\{x_1,x_2\}$ denotes a one- or a two-element set.

We will use the concept of a *function* from one set into another in the intuitive sense, meaning f is a function from X into Y ($f:X\to Y$) if for every element $x\in X$ there exists exactly one element $y\in Y$ assigned to x by f. Usually the element y assigned to x is denoted by f(x) and is called "the *value* of f for an *argument* x".

The sets X and Y are called the *domain* and the *range* of f, respectively.

The Story of Numbers

The story of numbers begins with positive integers. An educated hunter-gatherer returning to his cave would say to his cavewoman "I have killed two antelopes" or "3 rabbits" and she could reply "I have given birth to 4 babies in the meantime, so you better go and kill something bigger". These numbers (1,2,3 etc.) are called "natural" numbers for obvious reason. The set of all natural numbers is denoted by $\mathbb{N}$. The concept of a number evolved to include negative integers (resulting in $\mathbb{Z}$, the set of all integers) and fractions, resulting in $\mathbb{Q}$, the set all rational numbers. The ancient Greeks (Pythagoreans) believed that that was it until about the 5th century BC, when Hippasus was able to deduce that no rational number is the length of the diagonal of the unit square (according to the legend, he was subsequently murdered for shattering their beautiful theory). Somewhere along the way people also invented the number *zero*.

From today’s perspective, we look at the history of numbers as the constant effort to create more “complete” system in the sense that they include solutions to more and more types of equations – usually, but not exclusively, polynomial equations.

When it comes to solving polynomial equations of degrees higher than 1 the problem becomes more complicated. Even the construction of the set of real numbers $\mathbb{R}$ (let us just say "all possible distances and their negatives") was not enough to ensure solvability of all polynomial equations. We can easily construct a non-solvable polynomial equation with integer coefficients and the degree as small as 2, for example x^2+1=0. It turned out that the solution is fairly simple. It is enough to admit the existence of just one more object, the *imaginary unit i* with the property $i^2 = -1$ and all polynomial equations become solvable. Of course, if we want to create a consistent arithmetic system, admitting the number *i*, we must also accept all the consequences, i.e. all multiplicities of *i* and sums of real numbers and multiplicities of *i*.

Complex Numbers

The set of complex numbers is the set $\mathbb{C}$ of all expressions of the form a+b*i*, where a and b are real numbers and *i* is the *imaginary unit* satisfying the condition $i^2 = -1$.
Symbolically, $\mathbb{C}$ ={a+b*i* | a,b ∈ $\mathbb{R}$}. For a complex number *z*=a+b*i*, the two real numbers a and b are referred to as the *real part*, *Re*z, and the *imaginary part*, *Im*z, of *z*, respectively. So we can also write *z* = Re *z* + *i*Im *z*.

The question "where the hell is this *i* number on the real axis $\mathbb{R}$?" has as much sense as "where the hell is $\sqrt{2}$ in the set of quotients of integers?", i.e. none. The answer in both cases is obviously "nowhere".

Arithmetic of complex numbers

The usual *arithmetic operations* (addition, multiplication etc.) can be performed on complex numbers. They also conform to the properties of these operations for real numbers. To be more precise, we add and multiply complex numbers as if they were algebraic expressions with one extra rule: whenever i^2 appears it is replaced with -1. Hence,

$(a+bi) + (c+di) = (a+c) + (b+d)i$

and

$(a+bi) \cdot (c+di) = ac + adi + bic + bdi^2 = (ac-bd) + (ad+bc)i.$

Definition.
A *binary* (meaning two-argument) *algebraic operation* on a set X is any function $f: X \times X \to X$. The pair (X,#) is then called and *algebraic system* or, simply, an *algebra*.

Addition and multiplication are algebraic operations on ℂ while subtraction is NOT an operation on ℕ because it may result in a negative integer, not belonging to ℕ.
The important point in the definition, often overlooked, is that the result of the operation defined on a set must belong to the set.

We tend to use symbols like +,*,×,· for operations, and we place the operation symbol between the arguments rather than in front of the pair of arguments. +(a,b) seems an unnatural way of writing a+b.

Properties of operations

Definition.

Suppose (X,#) is an algebra. We say that

- # is *commutative* iff $(\forall a, b \in X) a\#b = b\#a$
- # is *associative* iff $(\forall a, b, c \in X)(a\#b)\#c = a\#(b\#c)$
- e is *an identity element* for # iff $(\forall a \in X) a\#e = e\#a = a$
- p is *an inverse element* for a iff $a\#p = p\#a = e$ (assuming that e is an identity element for #. If there is none, the question of an inverse for an element of X is meaningless).

Definition.

Suppose # and $ are (binary) operations on a set X, making (X,#,$) and algebra with two operations. We say that # is *distributive* with respect to $ (or distributive *over* $) iff $(\forall a, b, c \in X) a\#(b\$c) = (a\#b)\$(a\#c)$.

Replacing X with ℝ, # with · and $ with + we obtain the well-known principle of distributivity of multiplication of real numbers over addition: a·(b+c) = a·b+a·c.

It can be easily verified that both addition and multiplication of complex numbers are commutative and associative, and that multiplication is distributive over addition.

For example, let us verify associativity of multiplication:

LHS = $[(a+bi)(c+di)](e+fi)$ =
$[(ac-bd)+(ad+bc)i](e+fi)$ =
$(ace-bde-adf-bcf)+(acf-bdf+ade+bce)i$

while

RHS = $(a+bi)[(c+di)(e+fi)]$ =

$(a+bi)[(ce-df)+(cf+de)i]$ =

$(ace-adf-bcf-bde)+(acf+ade+bce-bdf)i$

so the expressions are identical. Distributivity of multiplication over addition can be verified in the same way.

Clearly, 0+0*i*, usually simply denoted by 0, is the only identity element for complex addition and 1+0*i* = 1 is the identity for multiplication.

It is also clear that $(-a) + (-b)i$ serves as the inverse for $z=a+bi$ with respect to addition. What about invertibility of complex numbers with resp. to multiplication? A simple calculation reveals that $\frac{a}{a^2+b^2} + \frac{-bi}{a^2+b^2}$ is the inverse of z=a+bi with respect to multiplication (unless $a^2 + b^2 = 0$ which happens only if $z = 0$). We obtain that every nonzero complex number is invertible with respect to multiplication, just like in the set of real numbers. Denoting the inverse of w=c+di by $w^{-1} = \frac{1}{c+di}$ we can write complex fractions in the standard form of a complex number:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac-bd)+(bc+ad)i}{c^2+d^2} = \frac{ac-bd}{c^2+d^2} + \frac{bc+ad}{c^2+d^2}\,i$$

Definition.
With every complex number z = a+bi we associate the *conjugate* of z, denoted by $\bar{z}$ = $a - b$i.

Notice that $z \cdot \bar{z} = a^2 + b^2$ is always a (nonnegative) real number. What we did in the last transformation of $\frac{z}{w}$ is called the expansion of the fraction by the factor of $\overline{w}$. This guarantees that we get rid of the imaginary unit i in the denominator.

Definition.
The *absolute value* or *modulus* of a complex number z = a+bi is the number |z| = $\sqrt{a^2 + b^2}$.

Notice that $|z|^2 = z \cdot \overline{z}$.

Geometrical approach to complex numbers

We can look at complex numbers as simply ordered pairs of real numbers. The coefficient i in $a+bi$ is there only to tell us which of a and b should be considered the first, and which the second element of the ordered pair. From this point of view, complex numbers can be identified with points of the Cartesian plane (or vectors anchored at the origin). We call this "geometrical interpretation of complex numbers".

We can rewrite the definition of addition and multiplication of complex numbers in this language:

$(a,b)+(c,d) = (a+c,b+d)$ which corresponds to geometrical addition of vectors anchored at $(0,0)$ with endpoints at (a,b) and (c,d), respectively, and

$(a,b)\cdot(c,d) = (ac-bd,ad+bc)$ (geometrical meaning of this operation is more complicated).

Hence, we can look at the algebra of complex numbers as an extension of arithmetic from real numbers to pairs of real numbers.

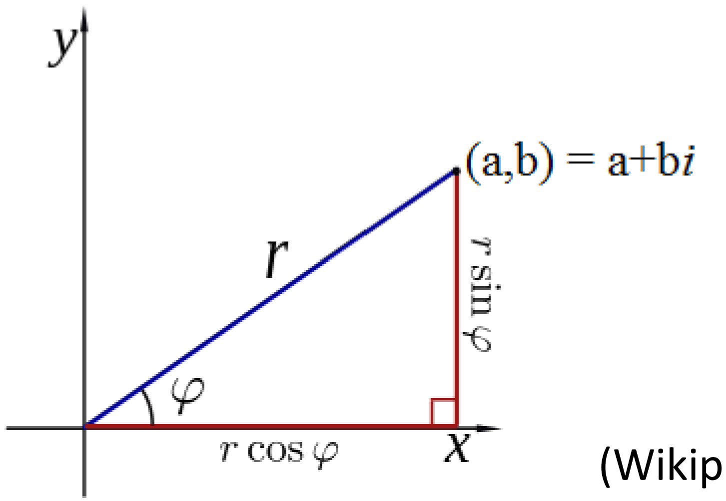
Notice that both complex addition and multiplication, when performed on complex numbers with imaginary parts are just "normal" arithmetic operations:
(a,0)(c,0) = (ac-0·0,a·0+0·c) = (ac,0). Or, in the standard form, (a+0i)(c+0i) = ac-0·0+(a0+0c)i = ac. The same for addition.

The complex modulus, when applied to a real number gives the "normal" absolute value: |a+0i| = $\sqrt{a^2 + 0^2}$ = $\sqrt{a^2}$ = |a|.

Comprehension test

1. The same happens for complex division.
2. $\overline{\mathrm{z}}$ = z if and only if z is a real number
3. $\overline{\overline{\mathrm{z}}} = \mathrm{z}$
4. $\overline{\mathrm{z} + \mathrm{w}} = \overline{\mathrm{z}} + \overline{\mathrm{w}}$
5. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$

A point *z* of the plane can be identified by its Cartesian coordinates, say (a,b), but also by its *polar coordinates*, i.e. the distance *r* from the origin and the angle α between positive half-axis OX and the segment (0,0)(a,b). Hence, (a,b)=(r*cos* α,r*sin* α) or, equivalently, z = a+b*i* = r(*cos* α + i*sin* α). Clearly, r =$\sqrt{a^2 + b^2}$, i.e. r = |z|



(Wikipedia & TT)

Definition.

The formula r(*cos* α + i*sin* α) is known as the *polar form* (sometimes *trigonometric form*) of a complex number z.